FINAL PROJECT – PORTFOLIO OPTIMIZATION FOR A SMALL INVESTOR AT VANGUARD WITH REQUIRED WITHDRAWALS

Aaron Koga

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This paper examines the risk-reward trade-off in the portfolio of an individual investor of Vanguard funds. The investor must pay for living expenses from the investments and conform to the minimum investment amount at Vanguard. The efficiency frontier is found for two situations: one where the investor makes investments and payments in dollars, and one where the investor makes investments in dollars and payments in yen.

I.   INTRODUCTION

A common question among investors is how and where to put their money in order to obtain maximum returns subject to certain requirements on risk and diversity. Often times, people not only want to earn a return on their money, but also want to use a portion of it for consumption. For example, retirees need to use their investments for personal consumption without totally depleting their investments. For most people, effective investing involves putting money into a combination of stocks, bonds, and money market funds or bank accounts. One convenient way to make and manage these different types of investments is with an investment company. One such company is Vanguard.

Vanguard is known for their low-cost, passively managed funds, which offer a wide array of investment options. Each fund has an associated minimum investment amount and annual fees.

II.   THE SITUATION AND PROBLEM

Fund Symbol (#)	Return/Yr (%)	Cost/Yr (%)	Min. Inv. ($)
VMMXX (1)	2.30	0.29	3,000
VFSTX (2)	3.74	0.21	3,000
VFINX (3)	6.07	0.18	3,000
VEIPX (4)	8.54	0.31	3,000
VSEQX (5)	14.59	0.35	3,000
NAESX (6)	12.99	0.23	3,000
VGTSX (7)	17.21	0.32	3,000

TABLE I: *Table showing Vanguard funds under consideration* [1].

Suppose an investor named Bob, who is currently working a modest job, is planning to go back to school to further his education in graduate school. He currently has $50,000 in savings, which he must use to cover his expenses for the next few years. Fortunately, he has received a small fellowship to support his graduate studies and thus anticipates that he will only need $500 a month from his savings. Bob has heard good things about Vanguard and wants to invest his $50,000 among a few funds, shown in TABLE I. Because Bob is going to be a graduate student, he will not have much time to tend to his investments. Therefore, he will only change his portfolio allocation once a quarter. Because he wants diversity, Bob does not want more than $15,000 in any one fund at any time. Bob wants to know the trade-off between the variance of his portfolio (risk) and his ending balance after one year of investment (reward).

Out of curiosity, Bob also wonders how an exchange rate would affect this optimization. If he were to study in Japan, his living expenses ($500/month) would be in yen. He wants to know if he could benefit from making smart decisions about when and how much money to exchange (from dollars to yen).

III.   FORMAL MODEL, NO EXCHANGE RATE

To address Bob's fist question (the one without the exchange rate), the following non-linear model was developed and optimized.

A.   Decision Variables

To allow quarterly changes to the portfolio, the decision variables were chosen as: $X_{i,j}$, the dollar amount to invest in each fund and $Y_{i,j}$, a binary variable linked to $X_{i,j}$. Here, $i = 1, 2, 3, 4$ denotes the quarter and $j = 1, 2, ..., 7$ denotes a specific fund. (note: there are 7 funds listed for consideration in TABLE I).

B.   Definitions

The return and cost of fund j, R_j and C_j respectively, are given in TABLE I. The return (given by the five year average, over the last five years) and cost for a fund were assumed to be constant over time. Thus the total quarterly return for fund j is given by $\frac{R_j - C_j}{4}$, and the value of an investment after a given quarter is

$$X_{i,j}\frac{1 + R_j - C_j}{4} = X_{i+1,j} \qquad (1)$$

The variance in any quarter is

$$var_i = \sum_{j=1}^{7} \sigma_j^2 P_{i,j}^2 + 2\sum_{j=1}^{6}\sum_{k=j+1}^{7} \sigma_{j,k} P_{i,j} P_{i,k} \qquad (2)$$

where σ_j is the variance of fund j and $\sigma_{j,k}$ is the covariance of funds j and k [2]. The values for σ_j and $\sigma_{j,k}$ were calculated using data for quarterly returns over the last five years. Also, $P_{i,j} = X_{i,j}/(\sum_{j=1}^{8} X_{i,j})$ is the portion of total money in fund j in quarter i. The total portfolio variance for the year is a simple average of the portfolio variance in each quarter:

$$Total\ Variance = \frac{\sum_{i=1}^{4} var_i}{4} \qquad (3)$$

S_i represents money reserved for consumption in a bank account. This surplus is given by:

$$S_i = \sum_j X_{i-1,j} \frac{1 + R_j - C_j}{4} - \sum_j X_{i,j} + (S_{i-1} - \$1,500) \qquad (4)$$

which is the difference between the money returned from last quarter's investments and money reinvested during this quarter, in addition to any left over surplus from last quarter.

C. Objective

To find the relationship between risk ($Total\ Variance$) and reward (the ending balance after one year) the ending balance of the portfolio was maximized, subject to a maximum $Total\ Variance$. In other words, the objective was

$$MAX: \quad \sum_j X_{4,j} \frac{1 + R_j + C_j}{4} \qquad (5)$$

D. Constraints

The usual non-negativity constraint,

$$X_{i,j} \geq \$0 \qquad (6)$$

was used. In addition, to provide the linking constraint and impose the minimum investment amount (\$3000),

$$X_{i,j} \geq \$3000 Y_{i,j} \qquad (7)$$

To impose the starting amount (\$50,000), the constraint

$$\sum_j X_{1,j} = \$50,000 \qquad (8)$$

was used. As specified above, because Bob does not want more than \$15,000 in any one fund,

$$X_{i,j} \leq \$15,000 \qquad (9)$$

was imposed. To satisfy the required withdrawals (\$500/month, which is \$1,500/quarter),

$$S_i \geq \$1,500 \qquad (10)$$

To trace out the efficiency frontier, a maximum value was set for the total variance.

$$Total\ Variance \leq A \qquad (11)$$

where A is a number. (Also see Appendix A for formulation.)

IV. SPREADSHEET IMPLEMENTATION, NO EXCHANGE RATE

This non-linear problem was implemented and solved in an excel spreadsheet. A screen capture of the spreadsheet is shown in FIG 3. Some formulas from the spreadsheet are highlighted in TABLE II. Excel's non-linear solver was used to maximize the ending balance for various values of the maximum $Total\ Variance$. The results, plotted in FIG 1, show that the maximum ending balance of \$50,900 occurs with a $Total\ Variance$ of $2.7(10^{-3})$. On the other hand, the minimum $Total\ Variance$ of $2.5(10^{-5})$ is achieved at the cost of a \$45,800 ending balance.
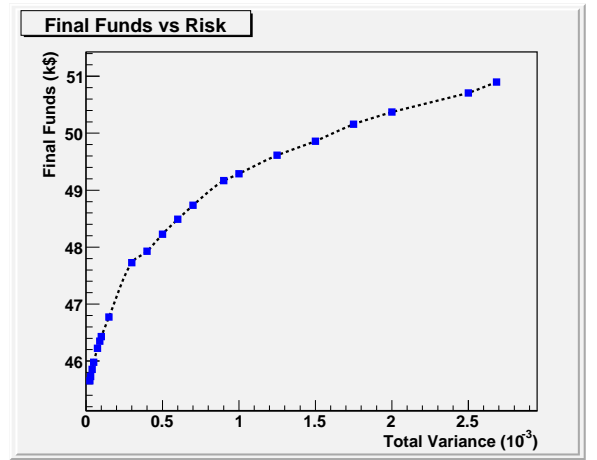

FIG. 1: *Plot of efficiency frontier with Final Funds vs Total Variance for the preceding model (no exchange rate).*

V. ADDING AN EXCHANGE RATE

The previous model assumes that Bob will be studying in the United States. However, if Bob were to study in Japan next year, he will be faced with living expenses in yen. Thus, he must take into account the dollar-yen exchange rate. Bob wants to know if he can save money by exchanging more dollars to yen at favorable exchange rates, rather than exchanging money without regard to the exchange rate. A few simple changes can be made to the previous non-linear model in order to answer this question.

Assume that the living expenses are now 178,410 yen/quarter (determined by the exchange rate on 4/7/07). To pay for this, money can be exchanged from dollars to yen once a quarter at the exchange rates of 117.36, 113.87, 114.52, and 117.49 yen/\$ [3].

Letting E_i be the exchange rate in quarter i, the equation governing the required withdrawals (EQ 10) becomes

$$S_i E_i \geq 178,410\ yen \qquad (12)$$

In addition to this,

$$S_i \leq \$6,000 \qquad (13)$$

sets a maximum limit on how much money can be exchanged at one time. Because Bob is not trying to use yen as an alternative investment, there is no reason to exchange more than $6,000 (his approximate total living expenses for the year) at a time. See Appendix B for the full formulation of this model.

This model was solved using an excel spreadsheet (a screen shot is shown in FIG 4). This spreadsheet was very similar to the one used to solve the model with no exchange rate. Again, excel's non-linear solver was used to maximize the ending balance for various values of the maximum *Total Variance*.

First, the spreadsheet was solved with $S_i E_i = 178,410$ *yen*; this simulated exchanging money without regard to the exchange rate (a modification to EQ 12). The spreadsheet was then solved with $S_i E_i \geq 178,410$ *yen*, simulating the exchange of money at opportune times.
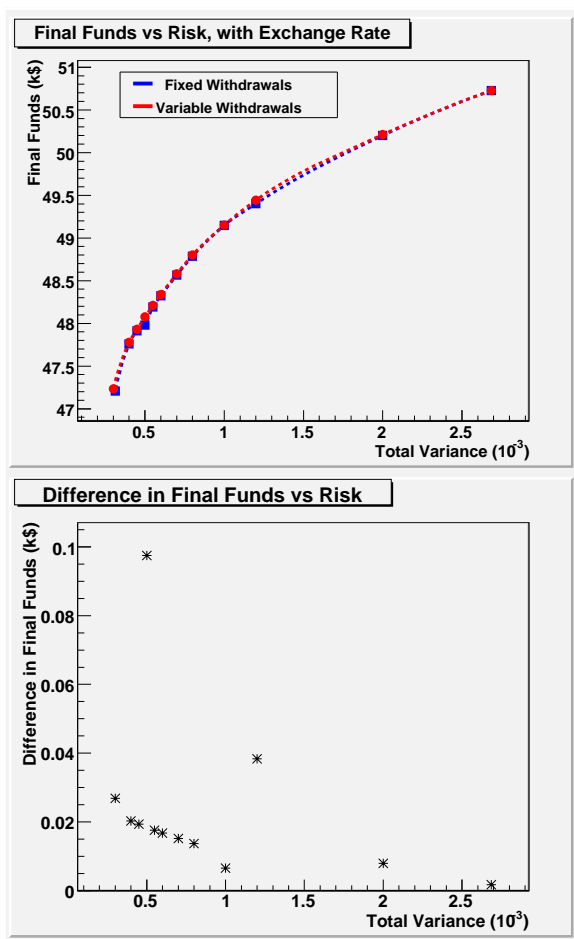

FIG. 2: *Plot of efficiency frontier with Final Funds vs Total Variance for the model taking into account the exchange rate (top). Also a plot of the money saved by exchanging money at opportune times (bottom).*

The results, plotted in FIG 2, show that the difference between exchanging money at opportune times does not make a big difference in the ending balance after one year. For the exchange rate used here, the largest difference is less than $100 and most of the savings is below $20. The ending balance as a function of the total variance in this

model with the exchange rate is very similar to the trade-off seen in the model without the exchange rate.

VI.   SUMMARY OF RESULTS

For an investor putting money a small number of Vanguard funds while using those funds to cover living expenses, there is a wide range of possible trade-offs between risk and reward. The ending balance after one year ranges from about $46,000 to almost $51,000. The portfolio variance ranges over two orders of magnitude, from $2.5(10^{-5}$ to $2.7(10^{-3})$.

If the investor wishes to live in Japan, he becomes subject to the fluctuating exchange rate. Even if money is exchanged from dollars to yen at favorable exchange rates rather than in regular amounts without regard for the exchange rate, no more than $100 can be saved. In fact, most savings is below $20. While meager savings is the result for the case presented here, it is possible that the savings can increase with different exchange rates (E_i). A more in-depth study using various exchange rates (perhaps generated by a monte carlo method) would provide a better answer for possible savings.

VII.   REFERENCES

[1] Data for Vanguard funds can be found at: https://flagship.vanguard.com/VGApp/hnw/FundsByType

[2] Ragsdale, C. T. *Spreadsheet Modeling and Decision Analysis, 5th Edition. Thomson, 2007.*

[3] Data for exchange rate can be found at: http://www.oanda.com/convert/fxhistory

VIII.   APPENDIX

A.   Model for No Exchange Rate

LET:
$X_{i,j}$ *be the dollar amount to invest in fund j in quarter i*
$Y_{i,j}$ *be a binary variable which is linked to* $X_{i,j}$.
R_j *be given and the annual return of fund j.*
C_j *be given and the annual cost of fund j.*
Total Variance=$\sum var_i$. where $var_i = \sum_{j=1}^{7} \sigma_j^2 P_{i,j}^2 +$
$2 \sum_{j=1}^{6} \sum_{k=j+1}^{7} \sigma_{j,k} P_{i,j} P_{i,k}$ *and σ_j is the variance of a fund j and $\sigma_{j,k}$ is the covariance of funds j and k.*
$P_{i,j}$ *be the portion of total money in fund j in quarter i.*
S_i *be surplus money, not invested in any fund during quarter i.*
BY CHANGING:
$X_{i,j}$, $Y_{i,j}$
MAX:
$\sum_j X_{4,j} \frac{1+R_j+C_j}{4}$
SUBJECT TO:
$X_{i,j} \geq 0$
$X_{i,j} \geq \$3000 Y_{i,j}$

$\sum_j X_{1,j} = \$50,000$
$X_{i,j} \leq \$15,000$
$S_i \geq \$1,500$
$Total\ Variance \leq A$, A is a number

B.   Model with Exchange Rate

LET:
$X_{i,j}$ be the dollar amount to invest in fund j in quarter i
$Y_{i,j}$ be a binary variable which is linked to $X_{i,j}$.
R_j be given and the annual return of fund j.
C_j be given and the annual cost of fund j.
$Total\ Variance = \sum var_i.$ where $var_i = \sum_{j=1}^{7} \sigma_j^2 P_{i,j}^2 +$

$2\sum_{j=1}^{6} \sum_{k=j+1}^{7} \sigma_{j,k} P_{i,j} P_{i,k}$ and σ_j is the variance of
a fund j and $\sigma_{j,k}$ is the covariance of funds j and k.

$P_{i,j}$ be the portion of total money in fund j in quarter i.
S_i be surplus money, not invested in any fund during
   quarter i.
E_i be the exchange rate in quarter i.
   BY CHANGING:
$X_{i,j}, Y_{i,j}$
   MAX:
$\sum_j X_{4,j} \frac{1+R_j+C_j}{4}$
   SUBJECT TO:
$X_{i,j} \geq 0$
$X_{i,j} \geq \$3000 Y_{i,j}$
$\sum_j X_{1,j} = \$50,000$
$X_{i,j} \leq \$15,000$
$S_i \geq (=)178,420\ yen$
$Total\ Variance \leq A$, A is a number
$S_i \leq \$6,000$

Investment	Inflow	Outflow	Amount	% Funds	Return	Cost	1	2	3	4	5		Binary	Min Order	Max Order	Max*Yi	Min*Yi
VMMXX	1	2	$ -	0.00%	2.30%	0.29%	-1	1.005025					0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	1	2	$ 15.00	30.00%	3.74%	0.21%	-1	1.008825					1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VFINX	1	2	$ -	0.00%	6.07%	0.18%	-1	1.014725					0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	1	2	$ -	0.00%	8.54%	0.31%	-1	1.020575					0	$ 3.00	$ 999.00	$ -	$ -
VSEQX	1	2	$ 15.00	30.00%	14.59%	0.35%	-1	1.0356					1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	1	2	$ -	0.00%	12.99%	0.23%	-1	1.0319					0	$ 3.00	$ 999.00	$ -	$ -
VGTSX	1	2	$ 15.00	30.00%	17.21%	0.32%	-1	1.042225					1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	2	3	$ -	0.00%	2.30%	0.29%		-1	1.005025				0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	2	3	$ 12.02	24.14%	3.74%	0.21%		-1	1.008825				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VFINX	2	3	$ 0.00	0.00%	6.07%	0.18%		-1	1.014725				0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	2	3	$ -	0.00%	8.54%	0.31%		-1	1.020575				0	$ 3.00	$ 999.00	$ -	$ -
VSEQX	2	3	$ 15.00	30.12%	14.59%	0.35%		-1	1.0356				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	2	3	$ 6.28	12.61%	12.99%	0.23%		-1	1.0319				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	2	3	$ 15.00	30.12%	17.21%	0.32%		-1	1.042225				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	3	4	$ -	0.00%	2.30%	0.29%			-1	1.005025			0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	3	4	$ 12.18	24.47%	3.74%	0.21%			-1	1.008825			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VFINX	3	4	$ -	0.00%	6.07%	0.18%			-1	1.014725			0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	3	4	$ 6.09	12.24%	8.54%	0.31%			-1	1.020575			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VSEQX	3	4	$ 15.00	30.14%	14.59%	0.35%			-1	1.0356			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	3	4	$ -	0.00%	12.99%	0.23%			-1	1.0319			0	$ 3.00	$ 999.00	$ -	$ -
VGTSX	3	4	$ 15.00	30.14%	17.21%	0.32%			-1	1.042225			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	4	5	$ -	0.00%	2.30%	0.29%				-1	1.005025		0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	4	5	$ 13.32	26.82%	3.74%	0.21%				-1	1.008825		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VFINX	4	5	$ -	0.00%	6.07%	0.18%				-1	1.014725		0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	4	5	$ -	0.00%	8.54%	0.31%				-1	1.020575		0	$ 3.00	$ 999.00	$ -	$ -
VSEQX	4	5	$ 15.00	30.20%	14.59%	0.35%				-1	1.0356		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	4	5	$ 4.85	9.76%	12.99%	0.23%				-1	1.0319		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	4	5	$ 15.00	30.20%	17.21%	0.32%				-1	1.042225		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00

	1	2	3	4	5
Total Funds:	$ 50.00	$ 49.80	$ 49.77	$ 49.67	$ 49.61
Variance:	0.002821	0.004462	0.003644	0.004073	0
Variance Weight:	1	1	1	1	1
Previous Surplus:	$ 5.00	$ 3.50	$ 0.00	$ (0.00)	$ (0.00)
Surplus Funds:	$ 1.50	$ 1.50	$ 1.50	$ 1.50	$ 49.61
Required Payment:	$ 1.50	$ 1.50	$ 1.50	$ 1.50	$ 1.50

Starting Amount: $ 50.00

Max In One Fund: $ 15.00

ENDING TOTAL: $ 49.61   0.00125001

TOTAL VARIANCE: 0.00125   0.00125

FIG. 3: *Screen capture of spreadsheet implementation in excel, no exchange rate.*

Cell(s)	Sample Formula	Purpose
H31:L31	sumif(H3:H30,-1,$D$3:$D:$30)+H35	Total Money
H32:L32	sumproduct(mmult('Fund History'!N92:T98,'Reg no Exchange'!$E3:$E9),'Reg no Exchange'!$E3:$E9)	Var_i
H35:L35	sumproduct($D$3:$D$30,I3:I30)+I34	S_i
E3:E30	D3/$H$31	$P_{i,j}$
D41	sumproduct(H33:L33,H32:L32)/12	Total Variance

TABLE II: *Some key formulas from the spreadsheet shown in FIG 3 (no exchange rate).*

Investment	Inflow	Outflow	Amount	% Funds	Return	Cost	1	2	3	4	5	Binary	Min Order	Max Order	Max*Yi	Min*Yi
VMMXX	1	2	$ -	0.00%	2.30%	0.29%	-1	1.005025				0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	1	2	$ -	0.00%	3.74%	0.21%	-1	1.008825				0	$ 3.00	$ 999.00	$ -	$ -
VFINX	1	2	$ -	0.00%	6.07%	0.18%	-1	1.014725				0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	1	2	$ 3.00	6.00%	8.54%	0.31%	-1	1.020575				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VSEQX	1	2	$ 15.00	30.00%	14.59%	0.35%	-1	1.0356				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	1	2	$ 15.00	30.00%	12.99%	0.23%	-1	1.0319				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	1	2	$ 15.00	30.00%	17.21%	0.32%	-1	1.042225				1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	2	3	$ -	0.00%	2.30%	0.29%		-1	1.005025			0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	2	3	$ -	0.00%	3.74%	0.21%		-1	1.008825			0	$ 3.00	$ 999.00	$ -	$ -
VFINX	2	3	$ -	0.00%	6.07%	0.18%		-1	1.014725			0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	2	3	$ 3.64	7.31%	8.54%	0.31%		-1	1.020575			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VSEQX	2	3	$ 15.00	30.18%	14.59%	0.35%		-1	1.0356			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	2	3	$ 15.00	30.18%	12.99%	0.23%		-1	1.0319			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	2	3	$ 15.00	30.18%	17.21%	0.32%		-1	1.042225			1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	3	4	$ -	0.00%	2.30%	0.29%			-1	1.005025		0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	3	4	$ -	0.00%	3.74%	0.21%			-1	1.008825		0	$ 3.00	$ 999.00	$ -	$ -
VFINX	3	4	$ -	0.00%	6.07%	0.18%			-1	1.014725		0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	3	4	$ 3.80	7.54%	8.54%	0.31%			-1	1.020575		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VSEQX	3	4	$ 15.00	29.79%	14.59%	0.35%			-1	1.0356		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	3	4	$ 15.00	29.79%	12.99%	0.23%			-1	1.0319		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	3	4	$ 15.00	29.79%	17.21%	0.32%			-1	1.042225		1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VMMXX	4	5	$ -	0.00%	2.30%	0.29%				-1	1.005025	0	$ 3.00	$ 999.00	$ -	$ -
VFSTX	4	5	$ -	0.00%	3.74%	0.21%				-1	1.008825	0	$ 3.00	$ 999.00	$ -	$ -
VFINX	4	5	$ -	0.00%	6.07%	0.18%				-1	1.014725	0	$ 3.00	$ 999.00	$ -	$ -
VEIPX	4	5	$ 4.00	7.92%	8.54%	0.31%				-1	1.020575	1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VSEQX	4	5	$ 15.00	29.69%	14.59%	0.35%				-1	1.0356	1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
NAESX	4	5	$ 15.00	29.69%	12.99%	0.23%				-1	1.0319	1	$ 3.00	$ 999.00	$ 999.00	$ 3.00
VGTSX	4	5	$ 15.00	29.69%	17.21%	0.32%				-1	1.042225	1	$ 3.00	$ 999.00	$ 999.00	$ 3.00

	1	2	3	4	5
Total Funds:	$ 50.00	$ 49.71	$ 50.36	$ 50.52	$ 50.73
Variance:	0.007996867	0.008218183	0.008039814	0.008027261	0
Variance Weight:	1	1	1	1	1
Previous Surplus:		(JPY 0.00)	(JPY 0.00)	(JPY 0.00)	
Surplus Funds:	$ 2.00	$ 1.07	$ 1.56	$ 1.52	$ 50.73
Exchange Rate:	117.36	56.31	114.51	117.49	50.73
Surplus Funds (YEN):	JPY 234.72	JPY 178.41	JPY 178.41	JPY 178.41	JPY 178.41
Required Payment (YEN):	JPY 178.41	JPY 178.41	JPY 178.41	JPY 178.41	JPY -

Starting Amount: $ 50.00 $ 50.00

Max In One Fund: $ 15.00

Max Surplus: $ 6.00

ENDING TOTAL: $ 50.73   0.00269018

TOTAL VARIANCE: 0.00269   1

FIG. 4: *Screen capture of spreadsheet implementation in excel, with exchange rate.*